2.5 The Chain Rule

Without the Chain Rule	With the Chain Rule
$y = x^2 - 1$	$y = \sqrt{x^2 - 1}$
$y = \sin x$	$y = \sin 5x$
$y = 3x - 2$	$y = (3x - 2)^7$
$y = x - \tan x$	$y = x - \tan x^2$

The Chain Rule

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

It may be helpful to think of the chain rule as "unpacking boxes." In doing so, you must "unpack" all the boxes as you get to them.

When using the Chain Rule, it is vitally important to rewrite if necessary so you can clearly identify the layers of the function.

Example 1:

Evaluate the following:

(a) $\frac{d}{dx}\left[\sqrt{x^2-1}\right]$

(b) $\frac{d}{dx}[\sin 5x]$

(c) $\frac{d}{dx}\left[(3x-2)^7\right]$

(d) $\frac{d}{dx}\left[x-\tan x^2\right]$

(e) $\frac{d}{dx}\left[\frac{1}{x+1}\right]$

(f) $\frac{d}{dx}\left[\sin^2 3x\right]$

(g) $\frac{d}{dx}\left[x\sec\left(x^2+x+1\right)\right]$

(h) $\frac{d}{dm}\left[\sec\left(4-\cos^3 5m\right)\right]$

(i) $\frac{d}{dx}\left[\left(\frac{3x-1}{x^2+3}\right)^2\right]$

Example 2:

Find all x-values on the graph of $f(x)=\sqrt[3]{\left(x^2-1\right)^2}$ for which $f'(x)=0$ and/or for which $f'(x)$ does not exist.

Example 3:

If $f(x) = \dfrac{1}{(1-2x)^3}$,

(a) Find the domain of $f(x)$.

(b) Show that the slope of every line tangent to the curve of $f(x)$ is positive. What conclusion about the behavior of the graph of $f(x)$ can you draw from this?

Example 4:

In each of the following, find $\frac{dy}{dx}$, then simplify $\frac{dy}{dx}$ by **factoring out the least powers**.

(a) $y = x^2\sqrt{1-x^2}$

(b) $f(x) = \frac{x}{\sqrt[3]{x^2+4}}$

Example 5:

For each function, rewrite if necessary, find $f'(x)$, then simplify.

(a) $f(x) = \cos 5x^2$

(b) $f(x) = (\cos 5)x^2$

(c) $f(x) = \cos(5x)^2$

(d) $f\left(x\right)=\cos^2 5x$

(e) $f\left(x\right)=\frac{1}{\sqrt{\cos 5x}}$

(f) $f\left(x\right)=\cos 5$

Example 6:

Find the equation of the tangent line to the graph of $f(x) = 2\sin x + \cos 2x$ at $x = \pi$.

Example 7:

Suppose that the differentiable functions f and g and their derivatives have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	–3
3	3	–4	2π	5

Evaluate the derivatives with respect to x of the following combinations at the given value of x.

(a) $2f\left(\frac{3x}{2}\right)$ at $x = 2$

(b) $f(x) + g(x)$ at $x = 3$

(c) $f(x) \cdot g(x)$ at $x = 3$

(d) $\frac{f(x)}{g(x)}$ at $x = 2$

(e) $f(g(x))$ at $x = 2$

(f) $\sqrt{f(x)}$ at $x = 2$

(g) $\frac{1}{g^2(x)}$ at $x=3$

(h) $\sqrt{f^2(x)+g^2(x)}$ at $x=2$