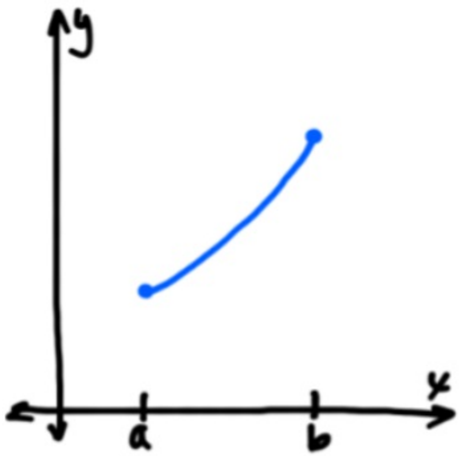
3.3 Increasing, Decreasing, and 1st Derivative Test

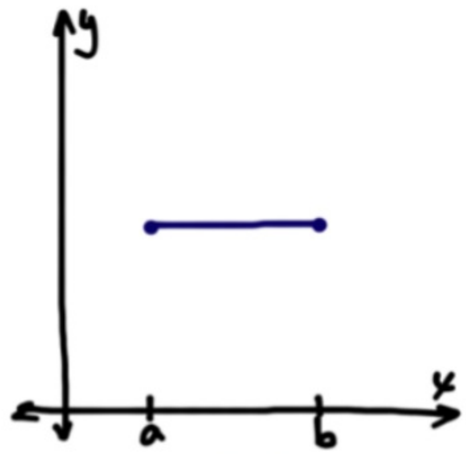
If a graph exists on an interval, it is doing one of three things:

1. Increasing (y-values rise as x-values increase)
2. Decreasing (y-values fall as x-values increase)
3. Staying Constant (y-values stay the same as x-values increase)



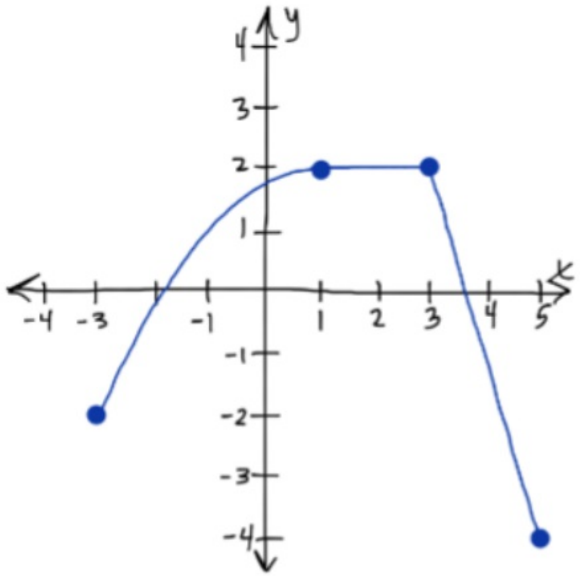
On (a,b),
f is increasing
$\Leftrightarrow$
$f'(x) > 0$

On (a,b),
f is decreasing
$\Leftrightarrow$
$f'(x) < 0$



On (a,b),
f is constant
$\Leftrightarrow$
$f'(x) = 0$

Example 1:
The graph of a function $f(x)$ defined on $[-3,5]$ is shown. List the **open** intervals over which the function is increasing, decreasing, and/or constant.



$f(x)$ is increasing on

$f(x)$ is decreasing on

$f(x)$ is constant on

Important Idea

If $f(x)$ is a **continuous function**, then $f'(x)$ can only change its sign at a **critical value**.

If $f(x)$ is a **discontinuous function**, then $f'(x)$ can change its sign either at a **critical value** or a **discontinuity.**

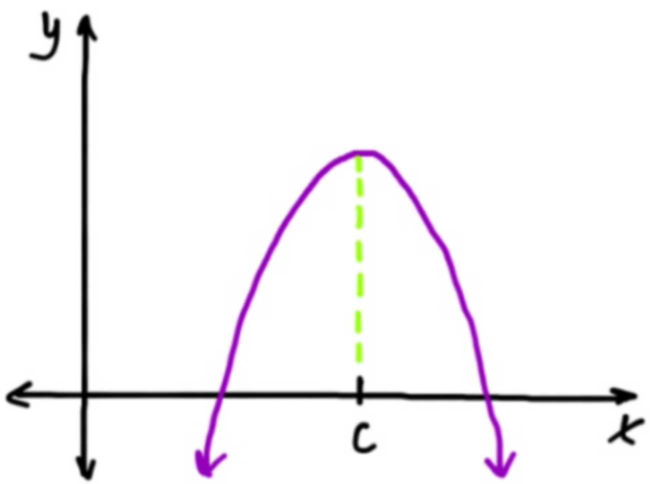
Example 2:

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing and/or decreasing. Justify.

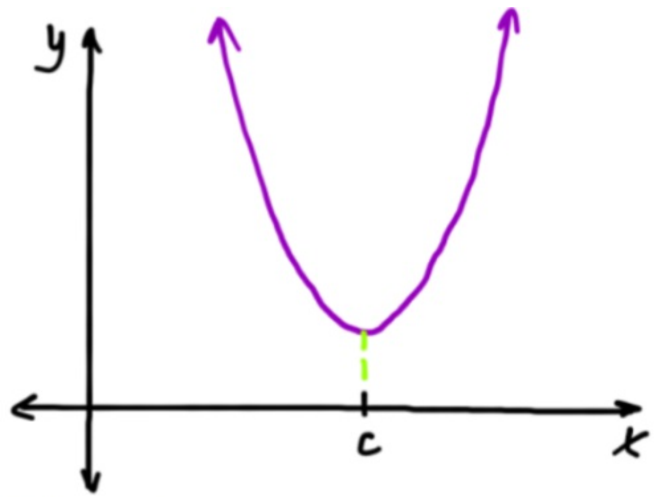
The First Derivative Test (for Relative Extrema of continuous functions)

Let $x = c$ be a critical value in the domain of a continuous function $f(x)$, then

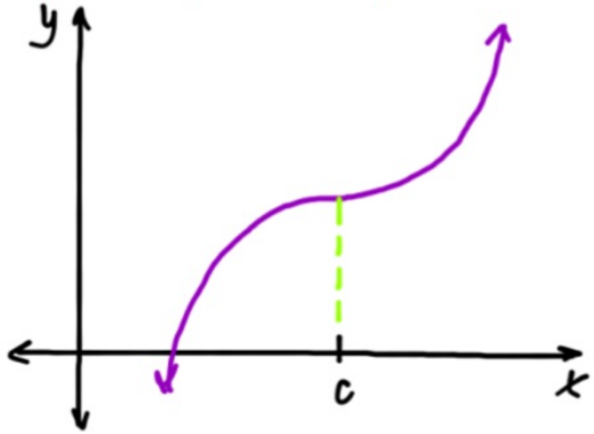
1. If $f'(x)$ changes from negative to positive at $x = c$, then f has a **relative minimum** at $x = c$ (or at $(c, f(c))$).
2. If $f'(x)$ changes from positive to negative at $x = c$, then f has a **relative maximum** at $x = c$ (or at $(c, f(c))$).
3. If $f'(x)$ is positive on both sides of $x = c$ or negative on both sides of $x = c$, then $f(c)$ is neither a relative maximum nor a relative maximum.



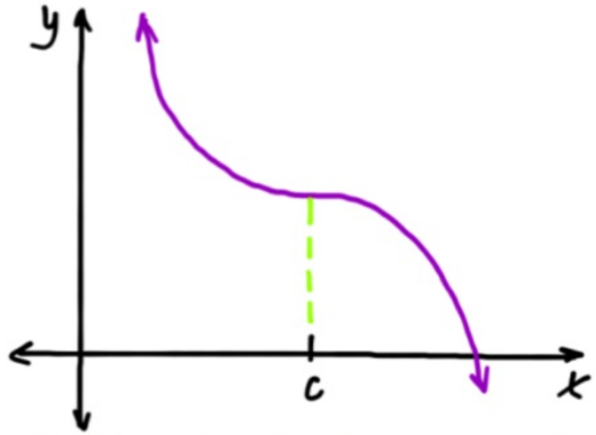
$f(x)$ has a local maximum at $x = c$, because $f'(x)$ changes from positive to negative at $x = c$.



$f(x)$ has a local minimum at $x = c$, because $f'(x)$ changes from negative to positive at $x = c$.



$f(x)$ has neither a local maximum nor a local minimum at $x = c$, because $f'(x)$ is positive immediately on either side of $x = c$.



$f(x)$ has neither a local maximum nor a local minimum at $x = c$, because $f'(x)$ is negative immediately on either side of $x = c$.

Example 3:

For $f(x) = \frac{2}{5}x^5 - \frac{7}{4}x^4 - \frac{4}{3}x^3 + 9$,

(a) Find the open intervals on which $f(x)$ is increasing and/or decreasing. Justify

(b) Determine the x-values of any local maximums or local minimums of $f(x)$. Justify.

Example 4:

Find the exact values of any relative extrema of the function $f(x)=\frac{1}{2}x-\sin x$ on the interval $[0,2\pi]$. Justify.

Example 5:

Find the relative extrema of $f(x)=\left(x^2-4\right)^{2/3}$. Justify.

Example 6:

Find the x-coordinates of the relative extrema of each of the following. Justify.

(a) $T(k)=\sqrt[3]{k^2}\,(2k-1)$

(b) $J(k)=\sqrt[3]{k}\,(2k-1)$

Remember that a local extrema may occur at a discontinuity, as long as the function is defined there. If a function is not continuous at a critical value, the first derivative test cannot be used.

Example 7:

Find the relative extrema of $f(x)=\begin{cases} x+3, & x\neq 1 \\ 2, & x=1 \end{cases}$. Justify.

Be careful of discontinuities where the function is NOT defined there, especially vertical asymptotes.

Example 8:

For both of the functions below, determine (i) the intervals of increasing/decreasing and (ii) the x-coordinates of the relative extrema. Justify.

(a) $f(x)=\frac{x^3+x}{x}$

(b) $f(x)=-\frac{x^4+1}{x^2}$