3.4 Concavity and 2nd Deriv. Test

Example 1:

Let $f(x)$ be a function such that $f'(x) > 0$ on some interval. Draw three different ways in which the graph of $f(x)$ might appear on this interval. In each case, analyze **how** the **slopes** of the graph of $f(x)$ are changing.

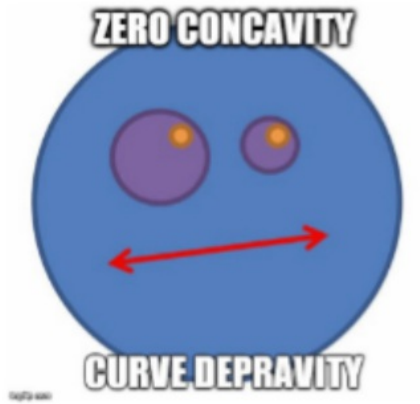
Important Idea: If f' **tells us how the** ***y*****-values of a graph of** ***f*** **are changing**, then the derivative of f', or f'', **tells us how the slopes of** f **are changing**.

Relation between f'' and concavity on an interval

For a continuous function $f(x)$ on an interval,

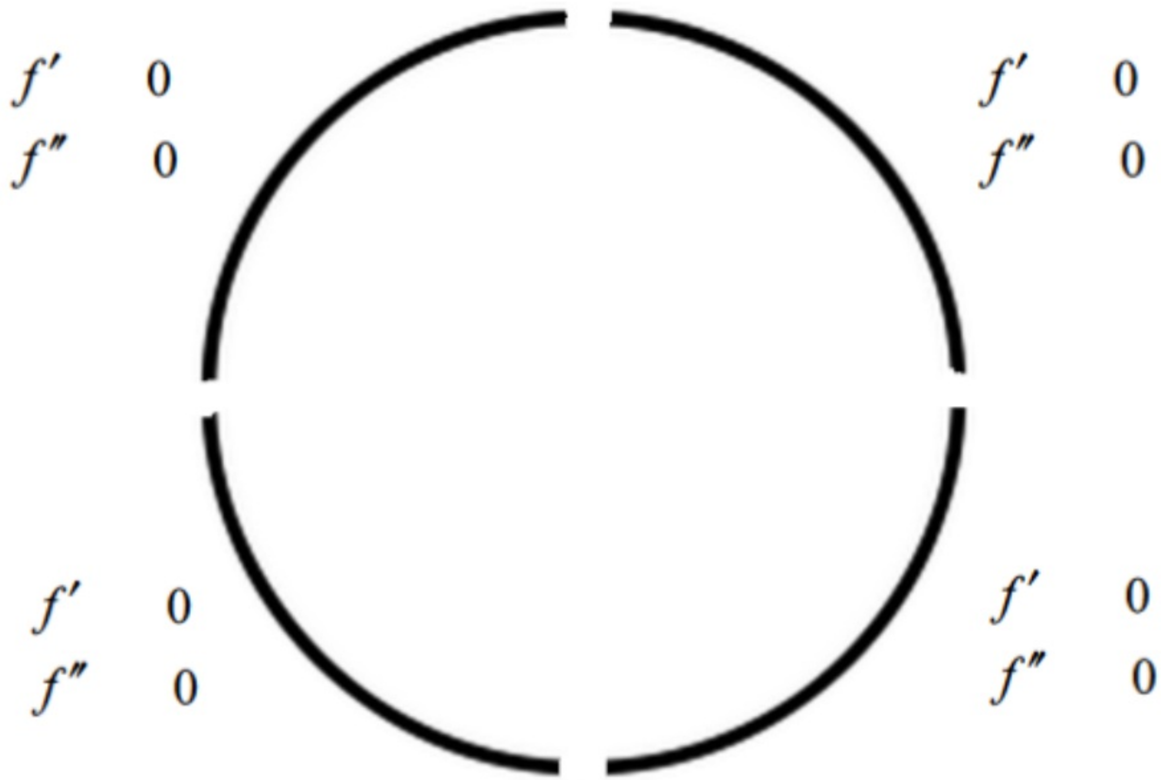
- $f''(x) > 0 \Leftrightarrow f$ is **concave up** (like a cup) on the interval $\Leftrightarrow$ slopes of f are increasing.
- $f''(x) < 0 \Leftrightarrow f$ is **concave down** (like a frown) on the interval $\Leftrightarrow$ slopes of f are decreasing.
- $f''(x) = 0 \Leftrightarrow f$ has no curvature/concavity $\Leftrightarrow$ slopes of f are constant.

Here's an easy way to remember the curvature of $f(x)$ based on the sign of $f''(x)$.



Example 2:

Determine the signs of f' and f'' for each of the curved segments below. Fill in the inequality.

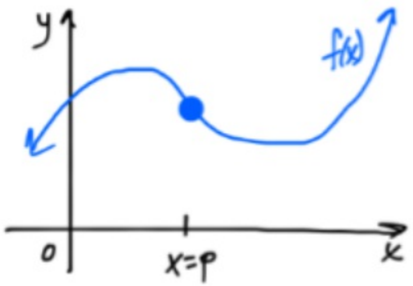


Definition

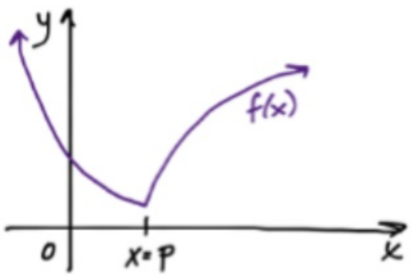
An x-value, $x = p$, in the domain of a function $f(x)$, is said to be an **inflection value** of $f(x)$ if the graph of $f(x)$ changes from either concave up to concave down at $x = p$ or from concave down to concave up at $x = p$.

That is, either f'' changes from positive to negative at $x = p$ or f'' changes from negative to positive at $x = p$.

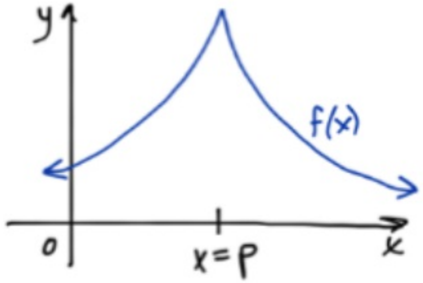
The point $(p, f(p))$ is called the **inflection point**.



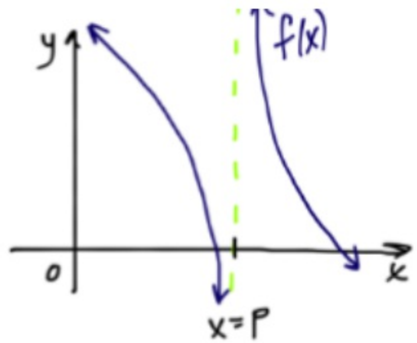
Inflection value at $x = p$



Inflection value at $x = p$



NO inflection value at $x = p$



NO inflection value at $x = p$

Definition

A **possible inflection value**, p.i.v., of a function $f(x)$, is an x-value **in the domain** of $f(x)$ such that either $f''(x)=0$ or $f''(x)=DNE$. These values are essentially critical values of f'.

Theorems

- The graph of a function $f(x)$ can change its concavity at either a possible inflection value or a discontinuity.
- An inflection value can only occur at a possible inflection value, p.i.v., but not every p.i.v. is an inflection value.

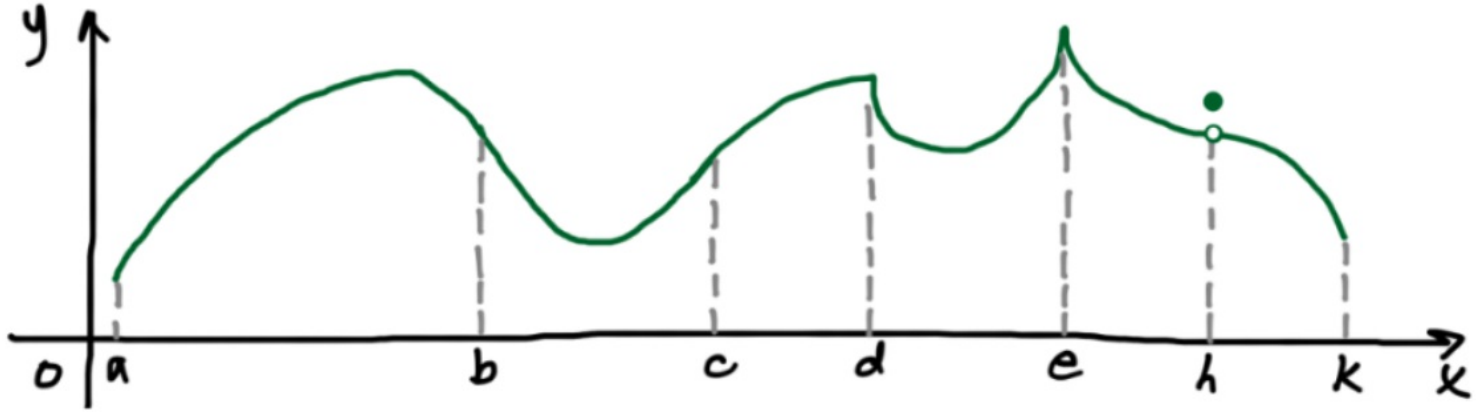
Test for intervals of concavity

1. Find any discontinuities of a function $f(x)$.
2. Identify any p.i.v.s in the domain of $f(x)$ by finding where $f''(x)=0$ or $f''(x)=DNE$
3. Using a number line chart, test the intervals by finding the sign of f'' on all intervals between any p.i.v.'s and discontinuities.
4. On the intervals where $f''(x)>0$, $f(x)$ is **concave up** (like a cup) on that interval. AND

   On the intervals where $f''(x)<0$, $f(x)$ is **concave down** (like a frown) on that interval.

Example 3:

The graph of a function $f(x)$ defined everywhere on the interval $[a,k]$ is given below.



(a) Give the possible inflection values, p.i.v.s

(b) At each p.i.v., determine if $f(x)$ has an inflection value. Justify.

(c) List the open intervals on which the graph $f(x)$ is concave up, concave down, and/or constant.

Important ideas to reiterate:

* Not every p.i.v. is an inflection value.
* Concavity can change at a discontinuity, such as a VA, but it won't be an inflection point.
* To find **possible inflection values** (p.i.v.'s), find any $c \in D_f$ such that $f''(c) = 0$ or $f''(c)$ is undefined at $x = c$ (as long as $f(c)$ is defined).
* A chart can help you efficiently test for concavity by determining the sign of f'' in between p.i.v.'s and **discontinuities**.

Example 4:
For each of the following functions, find (i) the p.i.v.s, (ii) the inflection values with justification, and (iii) the open intervals of concavity.

(a) $y = 3 + \sin x \quad x \in [0, 2\pi]$

(b) $f(x) = 6\left(x^2 + 3\right)^{-1}$

(c) $g(x) = \dfrac{x^2 + 1}{x^2 - 4}$

(d) $f(x) = x^4 - 4x^3$

(e) $y = e^{-x^2}$

(f) $g(x) = x^4$

(g) $f(x) = \sqrt[3]{x}$

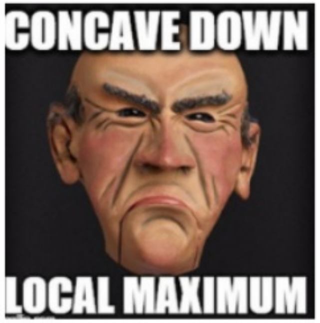
(h) $y = \begin{cases} \sqrt{-x}, & x \leq 0 \\ x^2, & x > 0 \end{cases}$

The Second Derivative Test (for Relative Extrema)

Let f be a function such that $x=c$ is a **critical value** of f such that $f'(c)=0$. If $f''(x)$ exists on an open interval containing $x=c$, then

1. If $f''(c)>0$, then $(c, f(c))$ is a relative minimum.

2. If $f''(c)<0$, then $(c, f(c))$ is a relative maximum.



3. If $f''(c)=0$, then the test fails, and the First Derivative Test must be used.

Example 5:

Find the relative extrema for $f(x) = -3x^5 + 5x^3$ using

(a) The First Derivative Test. Justify.

(b) The Second Derivative Test (if possible). Justify.

Example 6:

Selected values for a twice-differentiable function $f(x)$, continuous on , $-3 \le x \le 5$ is given below along with selected value for $f'(x)$, and $f''(x)$.

x	$f(x)$	$f'(x)$	$f''(x)$
-3	0	5	1
2	2	0	4
5	7	-1	0

(a) Is it safe to say that $f(x)$ is concave up on the interval $-3 < x < 5$? Why or why not?

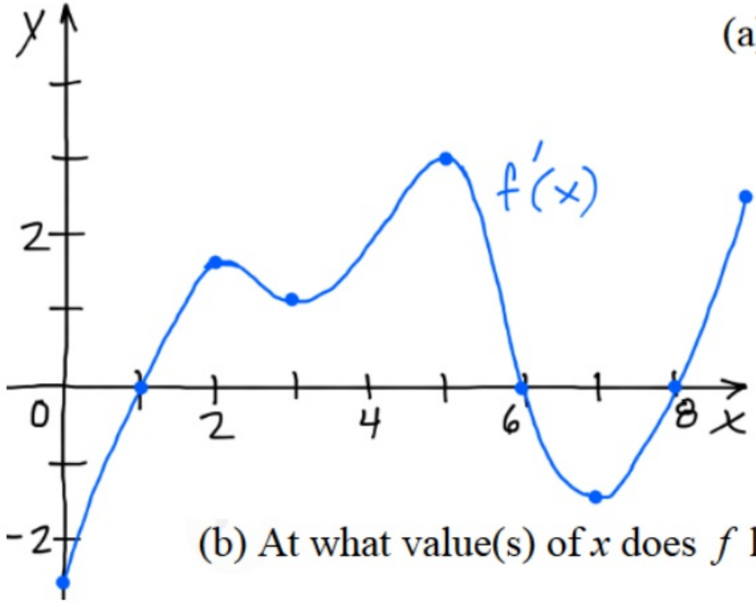
(b) (Review) Explain why there must be a $z \in (-3, 5)$ such that $f(z) = 6$.

(c) (Review) Explain why there must there must be a $w \in (-3, 5)$ such that $f'(w) = \frac{7}{8}$.

(d) Does $f(x)$ have a local maximum, local minimum, or neither at $x = 2$? Justify.

Example 7:

The graph of the derivative f' of a **continuous function** $f(x)$ on $[0,9]$ is shown below. Answer the following questions in complete sentences.



(a) On what open interval(s) is f decreasing? Justify.

(b) At what value(s) of x does f have a local maximum or minimum? Justify.

(c) On what intervals is f concave up? Justify.

(d) State the inflection values of f. Justify.

(e) Assuming that $f(0)=0$, sketch a graph of f. If possible, determine the x-value(s) at which f attains its maximum and/or minimum value(s).