4.2 Definite Integrals and Numeric Integration

Calculus answers two very important questions. The first, how to find the instantaneous rate of change, we answered with our study of the derivative. We are now ready to answer the second question: how to find the area of irregular regions.

We start by introducing sigma notation.

The sum S of n terms $a_1, a_2, a_{3, \ldots,} a_n$ is written as

$$S = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is called the **index of summation**, a_i is the ***i*th term** of the sum, and the **lower and upper bounds** of the summation are 1 and n.

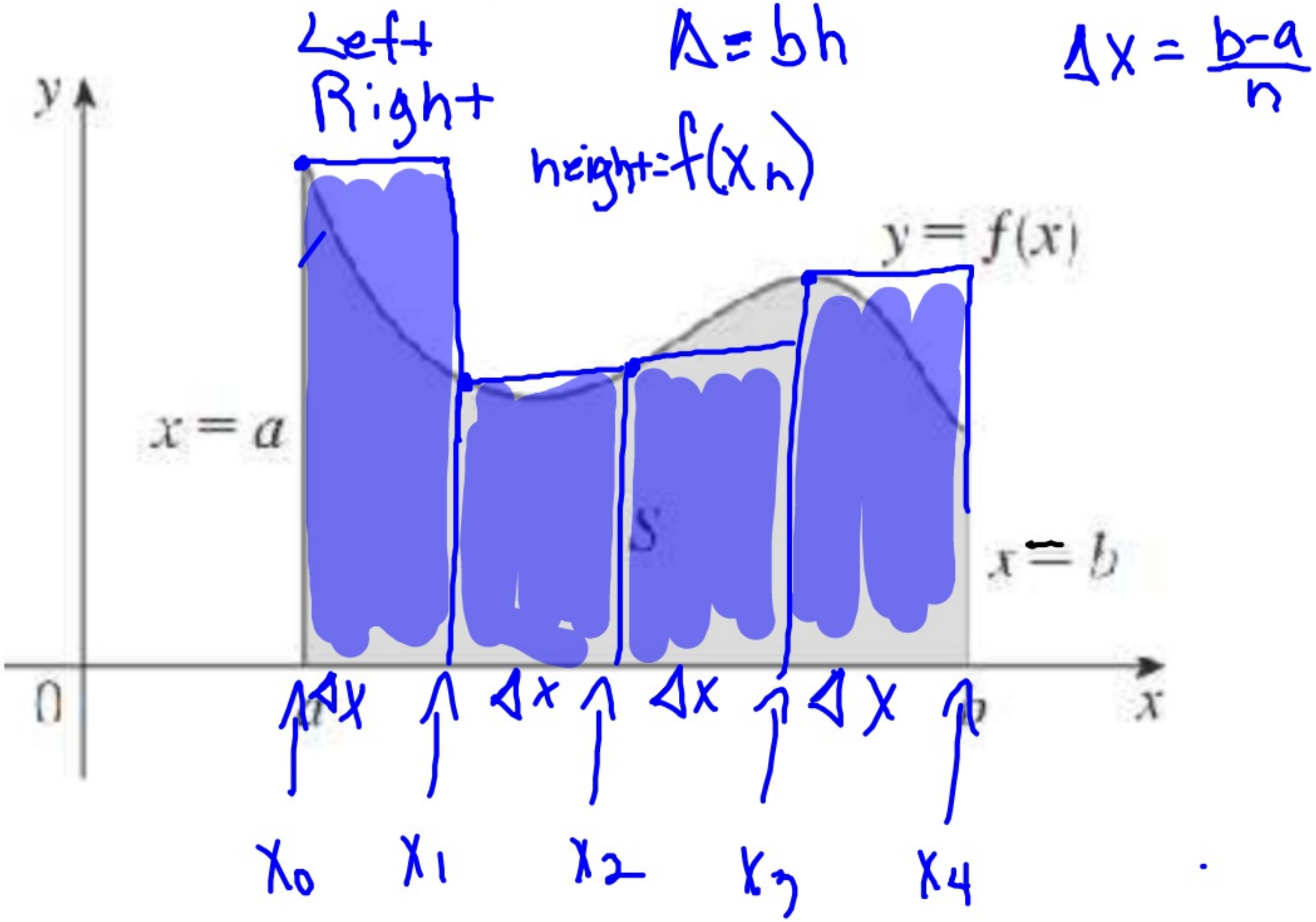
Example 1:

Evaluate a) $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$

b) $\sum_{n=0}^{5} \frac{x^{n+1}}{2^n} = \frac{x^{0+1}}{2^0} + \frac{x^{1+1}}{2^1} + \frac{x^{2+1}}{2^2} + \frac{x^{3+1}}{2^3}$

$+ \frac{x^{4+1}}{2^4} + \frac{x^{5+1}}{2^5} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{8} + \frac{x^5}{16} + \frac{x^6}{32}$

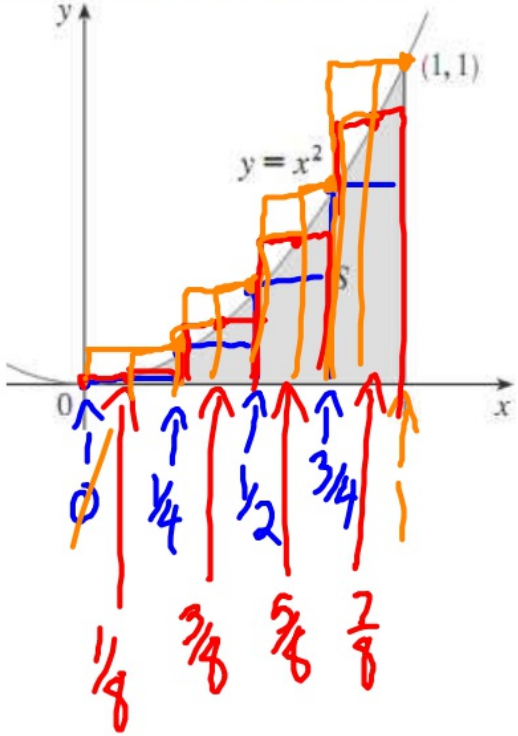
We will now approximate an irregular area bounded by a function, the x-axis between the vertical lines $x = a$ and $x = b$, like the one below, by finding the areas of many rectangles and summing them up.



Use 4 subintervals of equal width to approximate the area under the parabola $f(x)=x^2$ from $x=0$ to $x=1$, notated as region *S*, using the indicated method. Compare to the actual area using your calculator's numeric integration capabilities.

$\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{1-0}{4} = \frac{1}{4}$



(a) Rectangles using the left-endpoint, L_4

$A = \Delta x \cdot f(x_n)$

$A = \frac{1}{4} f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)$

$A = \frac{1}{4}\left[0^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2\right]$

$A = \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right] = \frac{1}{4}\left(\frac{14}{16}\right) = \boxed{\frac{14}{64}}$

(b) Rectangles using the right endpoint, R_4

$A = \frac{1}{4}\left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1)\right]$

$A = \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right]$

$A = \frac{1}{4}\left(\frac{30}{16}\right) = \boxed{\frac{30}{64}}$

$f(x) = x^2$

(c) Rectangles using the midpoint, M_4

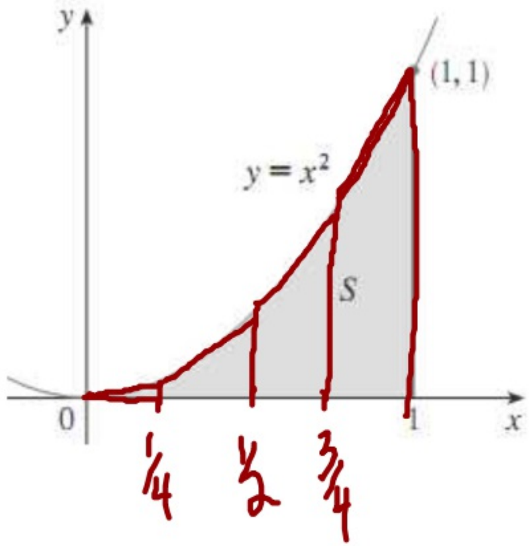
$$= \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{5}{8}\right)^2 + \left(\frac{7}{8}\right)^2\right] = \frac{1}{4}\left[\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right] = \frac{1}{4}\cdot\frac{84}{64}$$

$$= \frac{84}{256}$$

(d) Trapezoids, T_4

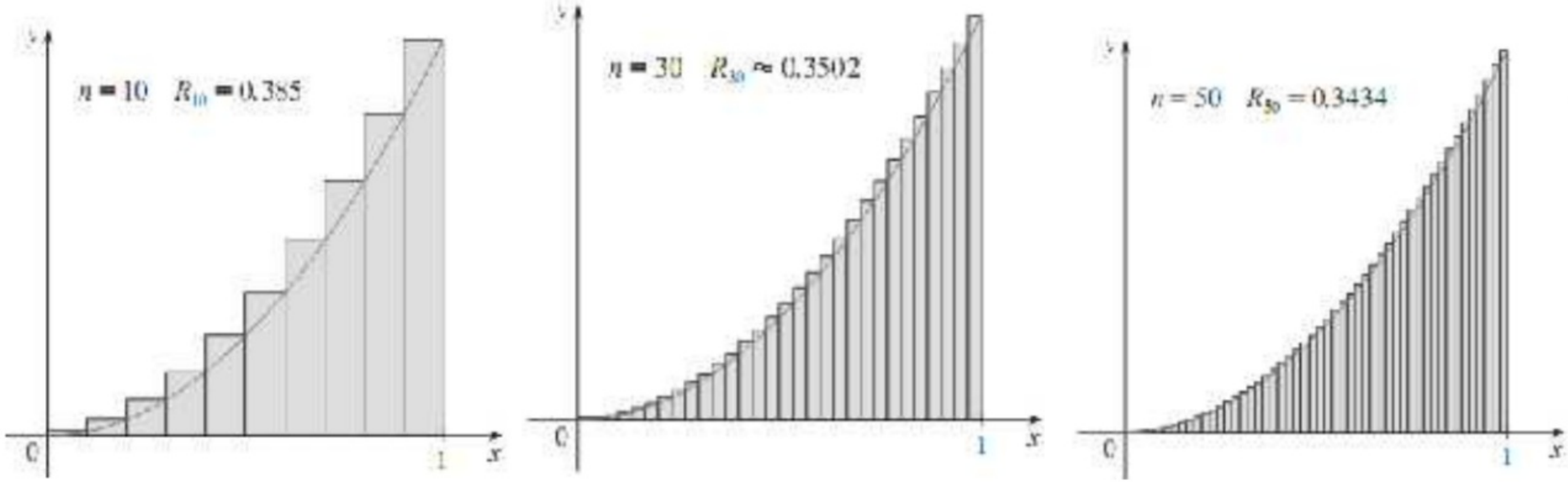
$$= \frac{22}{64}$$



(e) Your calculator's numeric integration capability.

$$= \frac{1}{3}$$

In this case, finding the area approximation using the left-endpoints of the intervals, L_4, gave us an under-approximation for the actual area. Using the right-endpoints, R_4, gives us an over-approximation. Together, these give us an upper and lower bound for the actual area (Note: depending on whether the function is increasing or decreasing, L_n or R_n could either be an upper or lower bound.)



n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3282500	0.3383500
1000	0.3328335	0.3338335

The process of finding the sum of the areas of rectangles to approximate area of a region is called **Riemann Sums**, after Bernhard Riemann, who pioneered the process.

Riemann proved that the finite process of adding up rectangular areas could be found by a routine analytic process know as **definite integration**. Here's the essence of his great, time-saving work.

$$A = \lim_{n \to \infty} \sum_{i=a}^{b} f\left(x_i\right) \Delta x = \int_a^b f\left(x\right) dx$$

Example 3:

Approximate the definite integral $\int_1^9 \sqrt{x}\, dx$ using 3 subintervals of equal width using each of the following methods. Determine if each approximation is an over or an under approximation:

$\Delta x = \frac{b-a}{n}$

$\Delta x = \frac{9-1}{3} = \frac{8}{3}$

(a) Left Riemann Sums

$A = 3[f(\ ) + f(\ ) + f(\ )]$

$A = 3[0 + \sqrt{3} + \sqrt{6}]$

$1 + \frac{8}{3}$

$\frac{3}{3} + \frac{8}{3} = \frac{11}{3}$

(b) Right Riemann Sums

(c) Trapezoids

Example 4:

Evaluate $\int_{0}^{4}(2x)\,dx$ by expressing the definite integral geometrically.

Area Under the Curve:

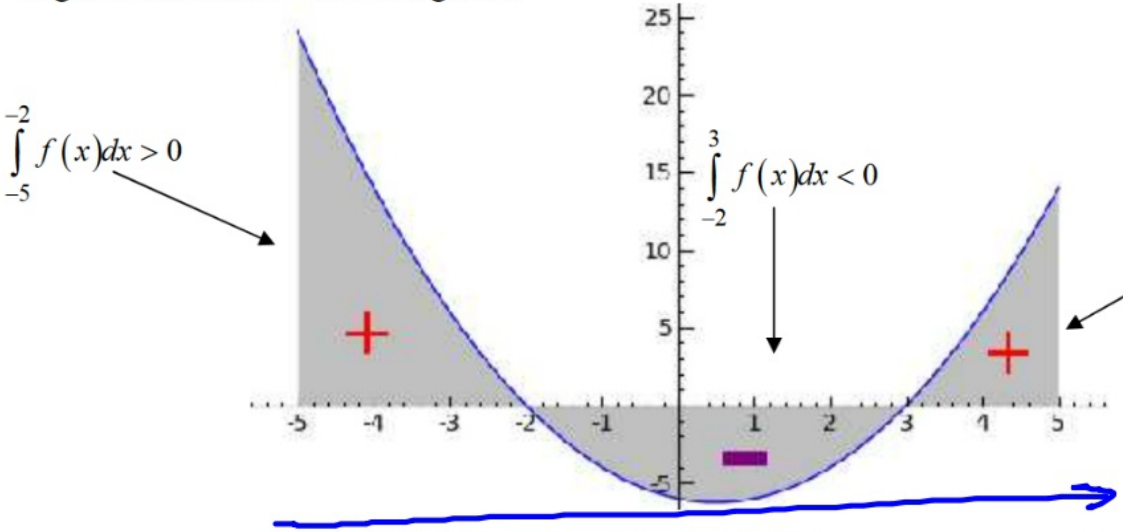
If $y = f(x)$ is **nonnegative** and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **from *a* to *b*** is the definite integral of *f* from *a* to *b*.

$$A = \int_a^b f(x)\,dx$$

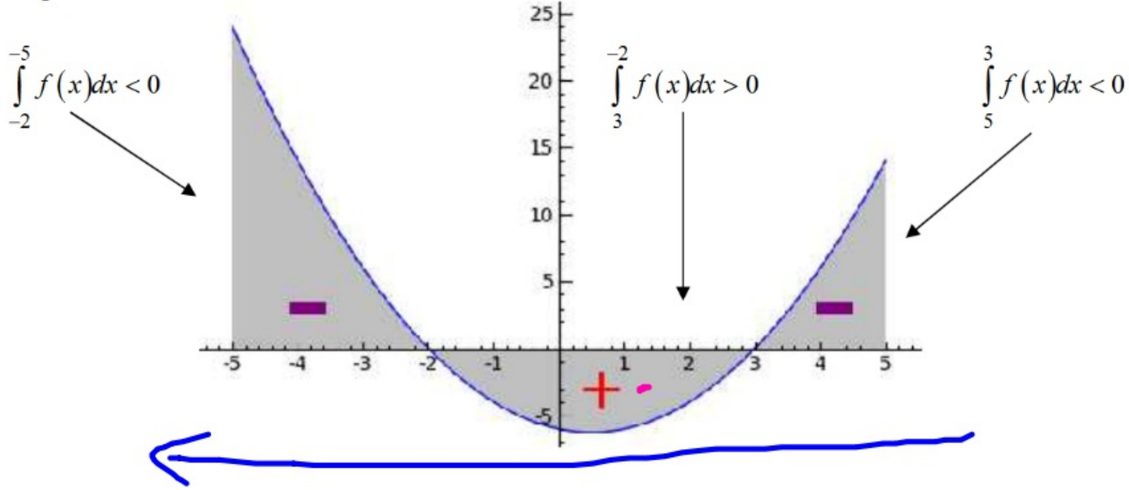
If $y = f(x)$ is **negative** and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **from *a* to *b*** is the **OPPOSITE** of the definite integral of *f* from *a* to *b*.

$$A = -\int_a^b f(x)\,dx$$

When integrating from **left to right** (chronological order), regions above the x-axis are positive and regions below the x-axis are negative.



When integrating from **right to left**, regions above the x-axis are negative and regions below the x-axis are positive.

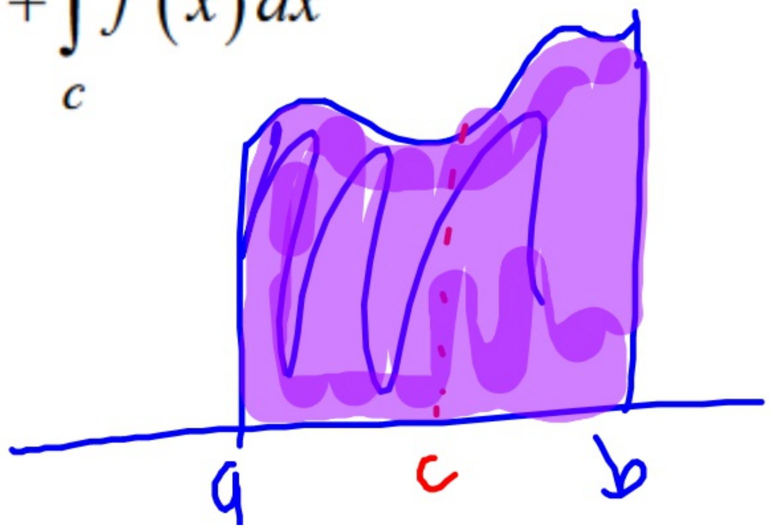


This second result can be summarized, in general, this way:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

AREA IS ALWAYS POSITIVE! AREA IS ALWAYS POSITIVE! AREA IS ALWAYS POSITIVE!

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Example 5: USING INTEGRALS TO FIND AREAS

If $f(x)=\begin{cases}3, & x<-1\\ 2-x, & x\geq -1\end{cases}$, write and evaluate an integral expression that gives the area of the region bounded by the graph of $f(x)$ and the x-axis on the interval $-3\leq x\leq 3$.

Theorem: Area of a region on a calculator

If $f(x)$ is a function defined on an interval $[a,b]$, the area of the region, A, bounded by $f(x)$ and the x-axis is given by

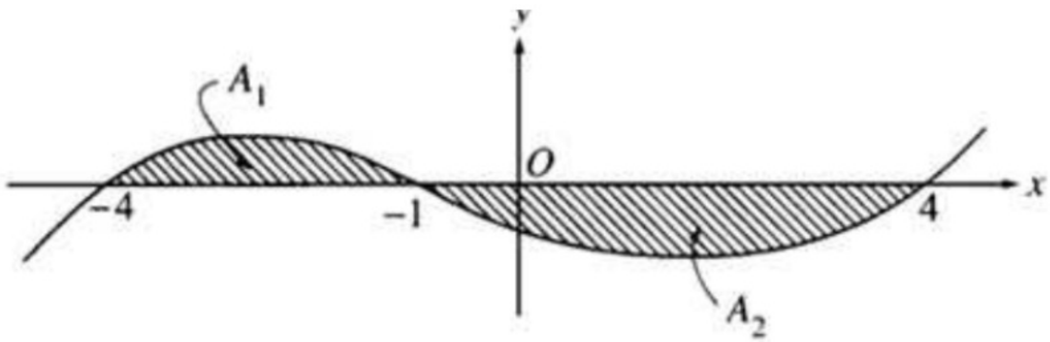
$$A=\int_a^b |f(x)|\,dx$$

Example 6:

Use your calculator to find the area of the region bounded by the graph of $f(x)=e^x-3$, the x-axis, and the vertical lines $x=0$ and $x=3$. Sketch and identify the region first.

Example 7: USING AREAS TO FIND INTEGRALS

The graph of $y = f(x)$ is shown below. If A_1 and A_2 are positive numbers that represent the areas of the shaded regions, then find, in terms of A_1 and A_2, the following:



$A_2 = -\#$

(a) $\int_{-4}^{-1} f(x)\,dx = A_1$

(b) $\int_{-1}^{4} f(x)\,dx = -A_2$

(c) $\int_{4}^{-1} f(x)\,dx = A_2$

(d) $\int_{-4}^{4} f(x)\,dx = A_1 - A_2$

(e) $\int_{-4}^{4} f(x)\,dx - 2\int_{-1}^{4} f(x)\,dx = A_1 - A_2 - 2(-A_2)$

$A_1 - A_2 + 2A_2$

$= A_1 + A_2$

Example 8

x		0	0.5	1	1.5	2	2.5	3
$f(x)$		2	4	6	7	4	1	5

$f(x)$ is a continuous function such that $f(x) \geq 0$ for all x. Selected values are given in the table above.

(a) Approximate $\int_0^3 f(x)\,dx$ using numeric methods as indicated by the data.

(b) Could any of these integral approximations represent approximations of the area of a region?

(c) Approximate $f'(1)$

a.) $L_6 = A = 0.5[2+4+6+7+4+1] = 0.5 \cdot 24 = 12$

$R_6 = A = 0.5[4+6+7+4+1+5] = 0.5 \cdot 27 = 13.5$

$M_3 = A = 1[4+7+1] = 12$

$T_6 = A = \frac{1}{2}\cdot\frac{1}{2}[2+2\cdot4+2\cdot6+2\cdot7+2\cdot4+2\cdot1+5] = \frac{1}{4}[51] = \frac{51}{4} = 12.75$

$\Delta \neq A$

c) $m = \frac{y_2-y_1}{x_2-x_1} = \frac{7-4}{1.5-0.5} = \frac{3}{1} = 3$

Ex 9)

x		0	1	3	6	6.6	8	10
$f(x)$		4	3	3	1	5	8	10

If $f(x)$ is a continuous function for all x, given selected values of f above,

(a) Approximate $\int_1^8 f(x)\,dx$ using numeric methods (reread the definite integral).

(b) Could any of these approximations represent the approximate areas of a region?

(c) Approximate $f'(7)$.

$A = \Delta x \cdot f(x_n)$

a)

$L_4 = A = 2(3) + 3(3) + .6(1) + 1.4(5) = 6 + 9 + .6 + 7 = 22.6$

$R_4 = A = 1.4(8) + .6(5) + 3(1) + 2(3) = 11.2 + 3 + 3 + 6 = 21.2$

c) $f'(7) = \frac{8-5}{8-6.6} = \frac{3}{1.4} = 2.14$

Properties of integrals:

1. If *f* is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$
2. If *f* is integrable on $[a,b]$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b c\,dx = c(b-a)$
4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Ex. 10

If $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} \left(3f(x)+2\right)dx$

$3\int_8^{10} f(x)\,dx + \int_8^{10} 2\,dx = 3\cdot 5 + 2(10-8)$

$= 15 + 4 = 19$

If $\int_4^{-6} f(x)\,dx = -3$, $\int_3^{7} f(x)\,dx = 9$, and $\int_4^{7} f(x)\,dx = 5$, find $\int_{-6}^{3} \left(\frac{f(x)}{2} - 3\right)dx$