4.4 Integration by u-sub and Pattern Recognition

Example 1:

Evaluate $\frac{d}{dx}\left[\tan\left(e^{4x^2}\right)\right] =$ $\sec^2\left(e^{4x^2}\right) e^{4x^2} \cdot 8x$

$= 8x \cdot e^{4x^2} \cdot \sec^2\left(e^{4x^2}\right)$

Example 2:

Evaluate $\int 8x \cdot e^{4x^2} \cdot \sec^2\left(e^{4x^2}\right) dx =$

Pattern Recogn.

$= \tan\left(e^{4x^2}\right) + C$

u-sub

$u = e^{4x^2}$

$dx \cdot \frac{du}{dx} = 8xe^{4x^2} dx$

$du = 8xe^{4x^2} dx$

$= \int \sec^2 u \, du$

$= \tan u + C$

$= \tan\left(e^{4x^2}\right) + C$

Deriv → Chain Rule

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

Integr. → Reverse Chain Rule

$$f\left(g\left(x\right)\right)+C=\int f'\left(g\left(x\right)\right)\cdot g'\left(x\right)dx$$

Antidifferentiation of a composite function

Let g be a function whose range is an interval, I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)dx=F\left(g\left(x\right)\right)+C$$

Evaluate $\int \left(x^2+1\right)^{12}(2x)\,dx$ using u-substitution as well as by pattern recognition.

$\frac{1}{13}(x^2+1)^{13}+C$

$u = x^2+1$

$du = 2x\,dx$

$\int u^{12}\,du = \frac{1}{13}u^{13}+C$

$= \frac{1}{13}(x^2+1)^{13}+C$

Example 4:

rider

Evaluate $\int 3x^3 \sin\left(2x^4+1\right)dx$ using u-substitution as well as by pattern recognition.

$-\frac{3}{8}\cos(2x^4+1)+C$

$\frac{3}{8}\int \sin u\,du$

$= -\frac{3}{8}\cos u + C$

$= -\frac{3}{4}\cos(2x^4+1)+C$

$\frac{d}{dx}\ -\frac{3}{8}\cos(2x^4+1)+C$

$\frac{3}{8}\sin(2x^4+1)\,8x^3 = 3x^3\sin(2x^4+1)$

When integrating by pattern recognition, you will collect no more than three different types of scalar/constant multiples out in front of your antiderivative:

- constant multiples that were there in the original integrand (I call these "riders"),
- constant multiples generated from an integration rule like the power rule (I call these "rule" constants), and finally,
- constant multiples that "correct" any unwanted constant multiple generated by the derivative of the "inside function." These values will always be the reciprocal of the unwanted value (I call these "corrections.")

After integrating, you can combine all of these scalar multiples to get your final answer. Oh, and don't forget your $+C$

Example 5:

Evaluate $\int \frac{(x+1)\sqrt[3]{x^2+2x}}{\pi}dx$

Example 6:

Each of the following have the same inside function, but a different outside function, and hence, a different rule of integration. Evaluate each.

(a) $\int 5\sin(2x)e^{\cos(2x)}dx$ (b) $\int 7\sin(2x)\sqrt[3]{\cos^2(2x)}dx$ (c) $\int 2\sin(2x)\cos^2(2x)\,dx$

(d) $\int (\sin 2x)\sin(\cos 2x)\,dx$

(e) $\int \frac{2\sin 2x}{3\sqrt[5]{\cos^2 x - \sin^2 x}}\,dx$

(f) $\int 11\sin(2x)\cos(2x)\,dx$

Example 7:

Evaluate $\int \sec^2 x \tan x dx$ two different ways. Show the antiderivatives are equivalent, but for a constant.

Example 8:
Evaluate

(a) $\int 4x^2 5^{x^3+7} \sec^2\left(5^{x^3+7}\right) dx$

(b) $\int 22x^2 \sin\left(5x^3\right) e^{\cos\left(5x^3\right)} dx$

We already know the following trig antiderivatives:

Example 9:
Evaluate the following:

(a) $\int \sin x dx$ (b) $\int \cos x dx$ (c) $\int \sec^2 x dx$ (d) $\int \csc^2 x dx$ (e) $\int \sec x \cdot \tan x dx$ (f) $\int \csc x \cdot \cot x dx$

We can expand our antidifferentiation repertoire by memorizing the integrals of the other four trig functions . . . but what ARE they?

Example 10:
Find each of the following by being clever, then memorize the results.

(a) $\int \tan x dx$ (b) $\int \cot x dx$ (c) $\int \sec x dx$ (d) $\int \csc x dx$

You can now handle integrals like the following . . .

Example 11:
Evaluate the following:

(a) $\int 5x^2 \tan\left(x^3+1\right) dx$ (b) $\int 2e^{-x} \sec\left(e^{-x}\right) dx$

Let's try some random integrals now to integrate all our integration methods.

Example 12:
Evaluate the following:

(a) $\int \frac{x}{x^2-4}dx$ (b) $\int \frac{5}{x\ln x}dx$ (c) $\int \frac{\sqrt[3]{\ln^2 x}}{4x}dx$ (e) $\int \frac{\csc(\ln x)}{ex}dx$ (d) $\int \frac{\arctan x}{1+x^2}dx$

Review the following antiderivatives:

(a) $\int \frac{dx}{\sqrt{1-x^2}} =$

(b) $\int \frac{dx}{1+x^2} =$

(c) $\int \frac{dx}{x\sqrt{x^2-1}} =$

Inverse Trig Integral forms: (MEMORIZE)

For some constant *a* and some function of *x*, *u* . . .

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} arc\sec\frac{|u|}{a} + C$$

Example 15:

Evaluate each of the following. Compare and Contrast.

(a) $\int \frac{dx}{\sqrt{4-x^2}}$ (b) $\int \frac{x}{\sqrt{4-x^2}}dx$ (c) $\int \frac{dx}{2+9x^2}$ (d) $\int \frac{x}{2+9x^2}dx$