5.5 L'Hopital's Rule

We're back to evaluating limits again. Recall the two **indeterminate forms** $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Example 1:
Make a guess as to what you think the limit will be, then check by evaluating the expression at the indicated values using your calculator.

a) $\lim_{x \to 0}\left(\frac{e^{3x}-1}{x}\right) = ?$

$\left.\frac{e^{3x}-1}{x}\right|_{x=0.1} =$ $\left.\frac{e^{3x}-1}{x}\right|_{x=0.01} =$ $\left.\frac{e^{3x}-1}{x}\right|_{x=0.001} =$

b) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=?$

$\left.\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)\right|_{x=1.1} =$ $\left.\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)\right|_{x=1.01} =$ $\left.\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)\right|_{x=1.001} =$

c) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=?$

$\left.\left(1+\frac{2}{x}\right)^x\right|_{x=100} =$ $\left.\left(1+\frac{2}{x}\right)^x\right|_{x=1000} =$ $\left.\left(1+\frac{2}{x}\right)^x\right|_{x=10000} =$

Let's review some earlier algebraic techniques for dealing with indeterminate forms.

Example 2:

a) $\lim\limits_{x \to -1} \dfrac{2x^2 - 2}{x+1} =$

b) $\lim\limits_{x \to \infty} \dfrac{3x^2 - 1}{2x^2 + 1} =$

c) $\lim\limits_{x \to 7} \dfrac{\sqrt{x+2} - 3}{x-7} =$

Not all indeterminate forms can be reconciled via algebraic techniques. Another method, discovered by Swiss mathematician Johann Bernoulli, is called **L'Hôpital's Rule**, named after the 17th century French mathematician Guillaume de L'Hôpital who did not discover it, but who first published it in the first-ever textbook on differential calculus. From the looks on their faces, can you guess which one is L'Hôpital and which one is Bernoulli?

L'Hôpital's Rule (or Bernoulli's Rule)

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ yields either of the indeterminate forms $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Example 3:

Evaluate the following.

a) $\lim_{x\to 0}\frac{e^{3x}-1}{x}=$

b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=$

c) $\lim_{x\to \frac{\pi}{2}}\frac{\sec x}{1+\tan x}=$

d) $\lim_{x\to \infty}\frac{\ln x}{2\sqrt{x}}=$

Sometimes we may need to repeat ourselves. Sometimes we may need to repeat ourselves.

Example 4:

a) $\lim_{x \to -\infty} \frac{x^2}{e^{-x}} =$

b) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} =$

Be careful when evaluating one-sided limits:

Example 5:

a) $\lim_{x \to 0^+} \frac{\sin x}{x^2} =$

b) $\lim_{x \to 0^-} \frac{\sin x}{x^2} =$

The rule works great, but it only works with the two forms $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$. There are other indeterminate forms including 0^0, 1^∞, $\infty-\infty$, $0\cdot\infty$, and ∞^0. We can still use the rule, but we have to first convert them to $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

Example 6:

a) $\lim_{x\to\infty} e^{-x}\sqrt{x} =$

b) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=$

Sometimes we need logs to come to our rescue.

Example 7:

a) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x =$

b) $\lim_{x\to 0^+} x^x =$

c) $\lim_{x\to\infty} x^{1/x} =$

Here's a fun one.

Example 8:

$$\lim_{x\to 1}\frac{\int_1^x \cos t\,dt}{x^2-1} =$$