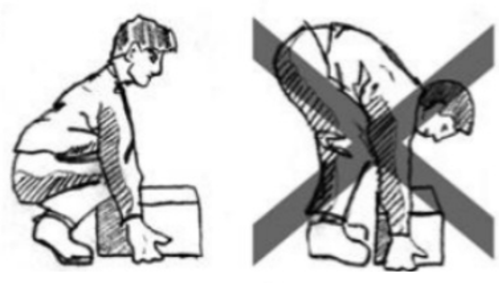
5.6 Improper Integrals

We've studied definite integrals this year, and, as you recall, they were all pretty routine, easy, and very courteous. Now we study improper integrals. As the name implies, they will be less routine, less easy, and perhaps lift boxes the wrong way.



Example 1:
Evaluate by using your calculator.

(a) $\int_1^{100} \frac{1}{x}\,dx =$ (b) $\int_1^{1000} \frac{1}{x}\,dx =$ (c) $\int_1^{1,000,000} \frac{1}{x}\,dx =$

Based on your values above, what do you think $\int_1^{\infty} \frac{1}{x}\,dx$ equals?

(d) $\int_{1}^{100} e^{-x}\,dx =$

(e) $\int_{1}^{1000} e^{-x}\,dx =$

What do you think $\int_{1}^{\infty} e^{-x}\,dx$ equals?

f) Sketch the graphs of $f(x)=\frac{1}{x}$ and $f(x)=e^{-x}$ in the first quadrants, and shade the regions represented by $\int_{1}^{\infty}\frac{1}{x}\,dx$ and $\int_{1}^{\infty} e^{-x}\,dx$.

Why would these two integrals give such different results?

How to deal with infinite intervals of integration:

1. If $\int_a^b f(x)dx$ exists of every $b > a$, then $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$, provided the limit exists as is finite.
2. If $\int_b^c f(x)dx$ exists of every $b < c$, then $\int_{-\infty}^c f(x)dx = \lim_{b\to-\infty} \int_b^c f(x)dx$, provided the limit exists as is finite.

Example 2:

(a) $\int_1^\infty \frac{1}{x^{3/2}} dx =$ (b) $\int_1^\infty \frac{1}{x^2} dx =$ (c) $\int_1^\infty \frac{1}{x^3} dx =$

Do you see a pattern as a function of the exponent??

Fact:

If $a > 0$, then $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$. These are called *p*-series integrals.

If $a = 1$ and $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ **converges** to $\frac{1}{p-1}$

Note: For those with a fancy for bathroom humor, it is interesting to note that the *p*-test relies so heavily on the number one. No elaboration needed.

Example 3:

(a) $\int_1^\infty \frac{1}{x^{2/3}}\,dx =$ (b) $\int_1^\infty \frac{1}{x^{1.1}}\,dx =$ (c) $\int_1^\infty x^{-7}\,dx =$ (d) $\int_1^\infty 3 \cdot x^{-3/2}\,dx$

Example 4:

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$$

Integrals such as $\int_{a}^{\infty} f(x)dx$, $\int_{-\infty}^{a} f(x)dx$, and $\int_{-\infty}^{\infty} f(x)dx$ are called **improper integrals**, not because they lift boxes incorrectly or even curse like a sailor but because of one of three reasons:

1. They have an infinite interval of integration.
2. They have a discontinuity on the interior of the interval of integration
3. Both 1) and 2).

They are evaluated by rewriting the integral as a proper integral and then using limits. Not every improper integral equals a finite number. In fact, you'd probably expect anything integrated to or from infinity will be infinite. An improper integral that equals a finite value is said to **converge** to that value. An improper integral that does not equal a finite number is said to **diverge**.

Improper integrals with an infinite interval of integration are easy to spot.

Example 5:

(a) $\int_{1}^{\infty} e^{-x} dx =$

(b) $\int_{-\infty}^{0} e^{x/4} dx =$

(c) $\int_{1}^{\infty} \frac{1}{x} dx =$

Example 6:

Let $f(x) = e^{-2x}$ for $0 \le x < \infty$, and let R be the unbounded region in the first quadrant below the graph of f. Find the volume of the solid generated when R is revolved around the x-axis.

Example 7:
"Rapid" fire . . . Watch out for indeterminate forms!!!!

(a) $\int_{1}^{\infty}(1-x)e^{-x}dx =$

(b) $\int_{0}^{\infty}e^{-x}\sin x dx =$

(c) $\int_{0}^{\infty}\frac{2dx}{x^2+4x+3} =$

Example 8:

Determine if the following converge or diverge. If they converge, find the value to which they converge.
This might be helpful:

Convergent plus Convergent = Convergent

Divergent plus Convergent = Divergent

Divergent plus Divergent = Divergent
$(\infty + \infty \text{ or } -\infty - \infty)$

Divergent – Divergent . . . . . . = Indeterminate
$(\infty - \infty \text{ or } -\infty + \infty)$

(a) $\displaystyle\int_1^\infty \frac{2+x}{x^2}\,dx$

(b) $\displaystyle\int_1^\infty \left[\frac{1}{x} - \frac{3x}{1+5x^2}\right]dx$