When an integral is improper has a finite interval of integration, it is improper because its interval spans an infinite discontinuity (vertical asymptote). These are harder to spot, so be vigilant!!

Example 12:

(a) Evaluate $\int_0^1 x^{-1/3}\,dx$ by evaluating the following on your calculator:

(i) $\int_{0.01}^1 x^{-1/3}\,dx$ (ii) $\int_{0.001}^1 x^{-1/3}\,dx$ (iii) $\int_{0.0001}^1 x^{-1/3}\,dx$

(b) Evaluate $\int_0^1 x^{-3}\,dx$ by evaluating the following on your calculator:

(i) $\int_{0.01}^1 x^{-3}\,dx$ (ii) $\int_{0.001}^1 x^{-3}\,dx$ (iii) $\int_{0.0001}^1 x^{-3}\,dx$

As you can see, if a function of the form $\frac{1}{x^p}$ tends to converge as $x \to \infty$ will tend to diverge as $x \to 0^+$ at the vertical asymptote. The exception, of course is $\frac{1}{x}$, which diverges in both directions.

When we recognize an infinite discontinuity at an endpoint, we have to set up a one-sided limit. When the infinite discontinuity is on the interior, we have to set up two integrals, approaching the VA from each side in each integral.

Example 13:
Verify your results from Example 12 by evaluating the following improper integrals.

(a) $\int_0^1 \frac{1}{x^{1/3}}\,dx =$

(b) $\int_0^1 \frac{1}{x^3}\,dx =$

Example 14:

(a) $\int_0^{27} \frac{dx}{\sqrt[3]{27-x}} =$

(b) $\int_0^{3} \frac{dx}{(x-1)^{2/3}} =$

Example 15:

What is the area of the region in the fourth quadrant bounded by the graphs of $y = 2\ln x$, $y = 0$, and $x = 0$?

Again, symmetry can be your friend . . .

Example 16:

Find the area bounded by the graph of $y = \frac{1}{x}$ and the x-axis over the interval $-2 \le x \le 2$.

Example 17:

Find the length of the curve defined by $y = \sqrt{4 - x^2}$ over the interval $0 \le x \le 2$. Use your result to find the circumference of the circle given by $x^2 + y^2 = 4$.

Sometimes an integral can be Dastardly Doubly Improper

Example 18:
Determine if the following integral is convergent or divergent. If it is convergent, find its value.

$$\int_0^{\infty} \frac{1}{x^2}\,dx$$

Example 19:

Let $f(x) = \frac{1}{x}$ for $1 \le x < \infty$, and let R be the unbounded region in the first quadrant below the graph of f.
Find the volume of the solid generated when R is revolved around the x-axis. (Note: The region is known as Gabriel's Horn or **Torricelli's Trumpet**.)