Unit 6: Differential Equations

6.1 Separable Differential Equations

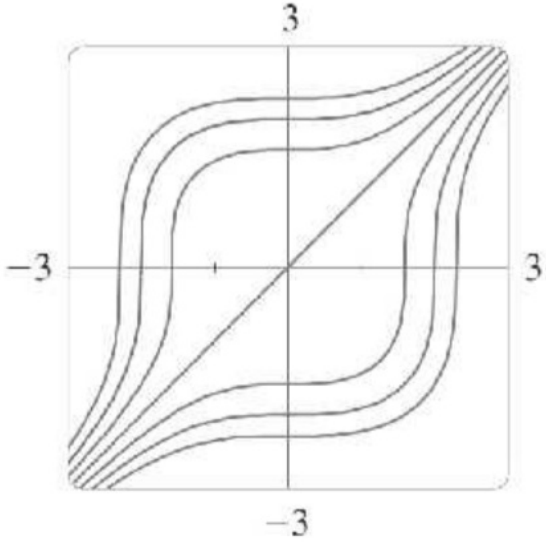
A **differential equation** is an equation that has one or more derivatives in it.

A **separable differential equation** is one in which all x and dx's can be separated from all the y and dy's.

For these types of problems, it is very, very, very important to SHOW THE SEPARATION OF THE VARIABLES.

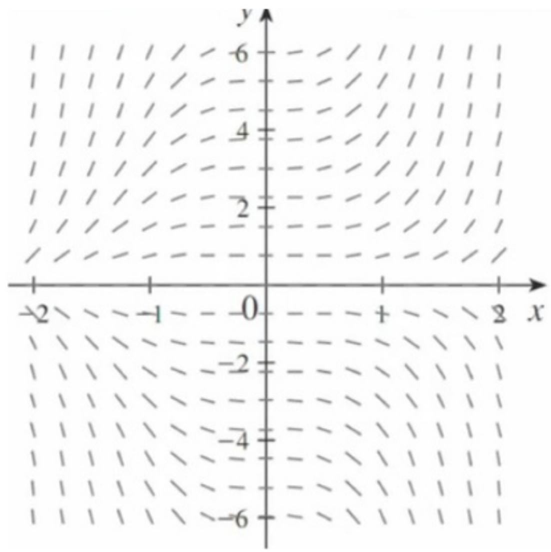
Example 1:

The graph of several solutions to the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ is shown. Solve the equation, then find the particular solution that satisfy the initial conditions (a) $y(0) = 2$, (b) $y(0) = -2$, and (c) $y(0) = 0$.



Example 2:

Find the general and particular solutions to the separable differential equation $\frac{dy}{dx} = x^2 y$ given the initial conditions (a) $f(0) = 1$ and (b) $f(0) = -2$.



Example 3:

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Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f, the slope is given by $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation of the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$ with the initial condition $f(1) = 4$.

(d) Use your solution from part (c) to find the exact value of $f(1.2)$.

(e) What would the solution equation be if the initial condition were $f(1) = -4$?

(a) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ $\left.\dfrac{dy}{dx}\right\rvert_{\substack{x=1\\y=4}} = \dfrac{3+1}{2\cdot 4} = \dfrac{4}{8} = \dfrac{1}{2}$	**1**: answer
(b) $y - 4 = \dfrac{1}{2}(x-1)$ $f(1.2) - 4 \approx \dfrac{1}{2}(1.2-1)$ $f(1.2) \approx 0.1 + 4 = 4.1$	**2** $\begin{cases} 1: & \text{equation of tangent line} \\ 1: & \text{uses equation to approximate } f(1.2) \end{cases}$

(c) $2y\,dy = (3x^2+1)\,dx$

$$\int 2y\,dy = \int (3x^2+1)\,dx$$

$y^2 = x^3 + x + C$

$4^2 = 1 + 1 + C$

$14 = C$

$y^2 = x^3 + x + 14$

$y = \sqrt{x^3+x+14}$ is branch with point $(1,4)$

$f(x) = \sqrt{x^3+x+14}$

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- 1: separates variables
- 1: antiderivative of dy term
- 1: antiderivative of dx term
- 1: uses $y = 4$ when $x = 1$ to pick one function out of a family of functions
- 1: solves for y

  0/1 if solving a linear equation in y
  0/1 if no constant of integration

Note: max 0/5 if no separation of variables

Note: max 1/5 [1-0-0-0-0] if substitutes value(s) for x, y, or dy/dx before antidifferentiation

(d) $f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$

1: answer, from student's solution to the given differential equation in (c)

Example 4:

For each of the following, find the general solution $y = f(x)$

by algebraically manipulating the differential equation first to the separable form $\frac{dy}{dx} = f(x)g(y)$

(a) $\frac{dy}{dx} = e^{x-y}$

(b) $\frac{dy}{dx} - x = xy^2$

Example 5:

For the differential equation $y\frac{dy}{dx} - x = 0$

(a) Find the general solution equation.

(b) Find the particular solutions that pass through (i) $(0,4)$ (ii) $(0,-4)$ (iii) $(-4,0)$ (iv) $(0,0)$