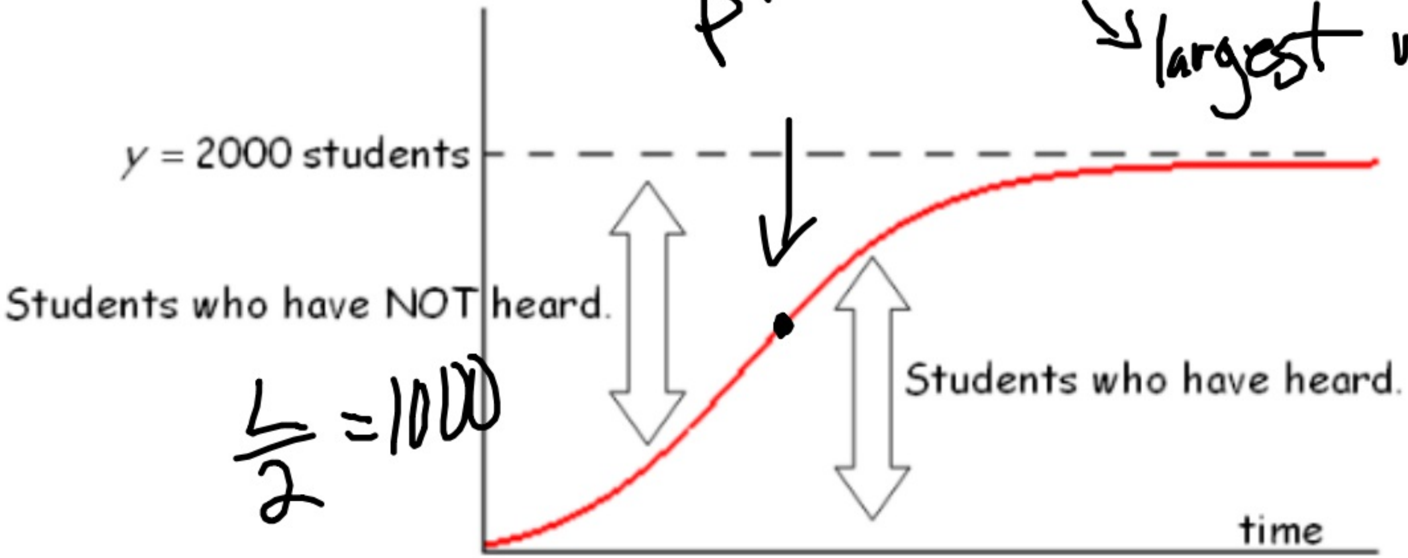
Imagine a rumor spreading throughout a school of 2000 students. The rate at which the rumor spreads is directly proportional to BOTH the students who have heard the rumor AND the students who have yet to hear the rumor as the number of people hearing the rumor approaches 2000.

The curve for the spread of the rumor might look like something shown below. This type of curve an growth is called **Logistic Growth**.

pt. of Inflation

largest rate of change



For quantities, y, that grow logistically with a carrying capacity of $y = L$, we can state the relation mathematically the following way:

$$\frac{dy}{dt} = ky(L - y)$$

Solving this differential equation requires an new integration technique called **Integration by Partial Fraction Decomposition**.

Example 1:

Add then simplify by finding a common denominator: $\frac{5}{2x-1} - \frac{2}{x-3}$

$$\frac{5(x-3) - 2(2x-1)}{(2x-1)(x-3)}$$

$$\frac{5x-15-4x+2}{(2x-1)(x-3)} = \frac{x-13}{(2x-1)(x-3)} = 2x^2$$

$$\boxed{\frac{x-13}{2x^2-7x+3}}$$

Example 2:

Evaluate $\int \frac{\boxed{x-13}}{2x^2-7x+3}dx$ $= \int \frac{5}{2x-1} - \frac{2}{x-3}\,dx$

$$\frac{\frac{1}{2}-13}{\frac{1}{2}-3} = \frac{-\frac{25}{2}}{-\frac{5}{2}} = 5$$

$$= \frac{1}{2}5\ln|2x-1| - 2\ln|x-3| + c$$

$$\frac{3-13}{(\ )\ 1\ 2\ 3} = \frac{-10}{5} = -2$$

Example 3:

Evaluate $\int \frac{x+5}{x^2+x-2}dx$

"Heaviside Cover-Up" Method

$$\int \frac{x+5}{(x+2)(x-1)}\,dx$$

$\boxed{-2}$ $\qquad \frac{-2+5}{-2-1} = \frac{3}{-3} = -1$

1 $\qquad \frac{1+5}{1+2} = \frac{6}{3} = 2$

$$\int \frac{-1}{x+2} + \frac{2}{x-1}\,dx$$

$$-\ln|x+2| + 2\ln|x-1| + C$$

$$2\ln|x-1| - \ln|x+2| + C$$

$$= \frac{\ln|x-1|^2}{\ln|x+2|} + C$$

of the numerator. When it's not we'll use our tried and true method of long dividing first.

Example 4:

Evaluate $\int \frac{3x^4+1}{x^2-1}dx$

$$\int 3x^2+3+\frac{4}{(x-1)(x+1)}\,dx$$

$$x^3+3x+\int \frac{2}{x-1}-\frac{2}{x+1}\,dx$$

1: $\frac{4}{1+1}=2$ $\quad$ -1: $\frac{4}{-1-1}=-2$

$$\begin{array}{r} 3x^2+3 \\ x^2-1\,\big)\,\overline{3x^4+1} \\ -(3x^4-3x^2) \\ \hline 3x^2+1 \\ -(3x^2-3) \\ \hline 4 \end{array}$$

$$x^3+3x+2\ln|x-1|-2\ln|x+1|+C$$

$$\boxed{x^3+3x+\ln\frac{|x-1|^2}{|x+1|^2}+C}$$

Back to Logistic Growth.

Example 5:

Solve the differential equation $\frac{dy}{dt} = ky\left(L - y\right)$

Logistic Growth

$$\text{If } \frac{dy}{dt} = ky\left(L - y\right), \text{ then } y = \frac{L}{1 + Ce^{-Lkt}}$$

Think "Lice Minus Licked." It's gross, but it works.

******Notice the prominence of our carrying capacity value L in each form of the equation. This is very important, especially when asked for the* ***limit at infinity*** *OR when asked to find the y-value when the y-values are* ***increasing most rapidly*** *(i.e. the inflection value: $y = \frac{L}{2}$).*

Example 6:

(Calculator) The population of Alaska since from 1900 to 2000 can be modeled by the following logistic equation.

$$P(t) = \frac{\boxed{895598}}{1 + 71.57e^{-0.0516t}}$$

2

where P is the population and t years after 1900, with $t = 0$ corresponding to 1900.

a) What is the predicted population of Alaska in 2020?
b) How fast was the population of Alaska changing in 1920? In 1940? In 1999?
c) When was Alaska growing the fastest, and what was the population then?
d) What information does the equation tell us about the population of Alaska in the long run?

b.) $P'(20) \approx 1640.258$

$P'(40) \approx$

$P'(99) \approx$

a.) 781218

c.) $\frac{L}{2} = \frac{895598}{2} = \boxed{447799}$

either the equation or the differential equation written in a DIFFERERNT FORMAT. This requires you to manipulate the equation to fit one of the two standard forms below:

$$\frac{dy}{dt} = ky(L - y) \rightarrow y = \frac{L}{1 + Ce^{-Lkt}}$$

Example 7:

The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $\frac{dP}{dt} = 0.008P(100 - P)$, where t measured in years.

a) What is the carrying capacity for bears in this wildlife preserve? 100

$L = 100$ $\quad \frac{L}{2} = \frac{100}{2} = 50$

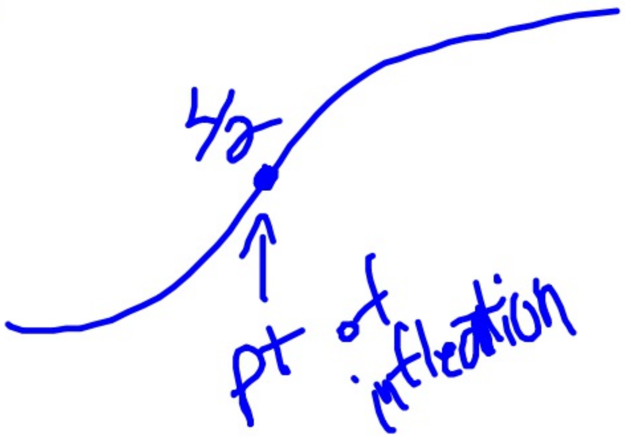
b) What is the bear population when the population is growing the fastest?

c) What is the rate of change of population when it is growing the fastest?

c) $P = 50$

$$\frac{dP}{dt} = 0.008(50)(100 - 50)$$

$$\boxed{\frac{dP}{dt} = 20}$$



Example 8:

Suppose that a population develops according to the ~~logistic differential~~ equation $\frac{dP}{dt} = 0.2P - 0.002P^2$,

where t is measured in weeks, $t \geq 0$.

$$\frac{dP}{dt} = .002P(100 - P)$$

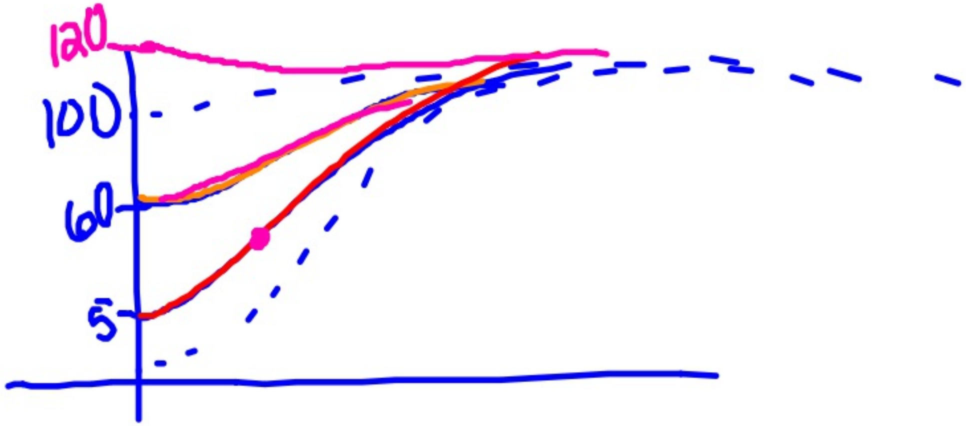
a) If $P(0) = 5$, what is $\lim_{t\to\infty} P(t)$? 100

b) If $P(0) = 60$, what is $\lim_{t\to\infty} P(t)$? 100

c) If $P(0) = 120$, what is $\lim_{t\to\infty} P(t)$? 100

d) Sketch the solution curves for a), b), and c). Which one has an inflection point?

d) $L = 100$



Example 4

The rate at which the flu spreads through a community is modeled by the logistic differential equation

$\frac{dP}{dt} = 0.001P(3000 - P)$, where t is measured in days, $t \geq 0$.

$S \neq S$

a) If $P(0) = 50$, solve for P as a function of t.

b) Use your solution to a) to find the size of the population when $t = 2$ days.

c) Use your solution to a) to find the number of days that have occurred when the flu is spreading the fastest.

$y = \frac{L}{1+Ce^{-Lkt}}$

a.) $y = \frac{3000}{1+Ce^{-3t}}$

$50 = \frac{3000}{1+Ce^{0}}$

$50 = \frac{3000}{1+C}$

$1+C = \frac{3000}{50}$

$1+C = 60$

$C = 59$

$P = \frac{3000}{1+59e^{-3t}}$

b.) $P = \frac{3000}{1+59e^{-3\cdot 2}}$ ppl

$P = \frac{3000}{1+59e^{-6}}$

$P = 2617.238$

c.) $1500 = \frac{3000}{1+59e^{-3t}}$

$$\int\left[\frac{1/L}{y}+\frac{1/L}{L-y}\right]dy = kt + C$$

$$\frac{1}{L}\int\left(\frac{1}{y}+\frac{1}{L-y}\right)dy = kt + C$$

$$\frac{1}{L}\left[\ln|y| - \ln|L-y|\right] = 1$$

$$y = \frac{L}{1+\frac{1}{Ce^{Lkt}}}$$

$$\boxed{y = \frac{L}{1+Ce^{-Lkt}}}$$

$$\left|\frac{y}{L-y}\right| = e^{(Lkt+C)}$$

$$\frac{y}{L-y} = Ce^{Lkt}$$

$$y = Ce^{Lkt}(L-y)$$

$$y = Ce^{Lkt} - yCe^{Lkt}$$

$$y(1+Ce^{Lkt}) = Ce^{Lkt}$$

$$y = \frac{Ce^{Lkt}}{1+Ce^{Lkt}}$$

20) = 1678.078 ppl/yr

40) = 4127.794 ppl/yr

(99) = 9741.747 ppl/yr

$(t) = \frac{895598}{2} = 447799$ ppl

$t = 82.765$ yrs

≈ 1982

Notes 7.5T

$$+3x+4\int\left[\frac{1}{(x-1)(x+1)}\right]dx$$

$$x^3+3x+4\int\left[\frac{1/2}{x-1}-\frac{1/2}{x+1}\right]dx$$

$$x^3+3x+2\int\left(\frac{1}{(x-1)}-\frac{1}{(x+1)}\right)dx$$

$$x^3+3x+2\left[\ln|x-1|-\ln|x+1|\right]+C$$

ential equation $\boxed{\frac{dP}{dt}} = 0.008P(100-P)$, where t measured in years.
$k \cdot P(L-P)$

What is the carrying capacity for bears in this wildlife preserve? 100 bears
What is the bear population when the population is growing the fastest? 50 bears
What is the rate of change of population when it is growing the fastest?

$$\left.\frac{dP}{dt}\right|_{P=50} = 0.008(50)(100-50) \text{ bears/yr}$$
$$= \boxed{20 \text{ bears/yr}}$$

xample 8:

$\frac{dP}{dt} = kP(L-P)$

ippose that a population develops according to the logistic differential equation $\frac{dP}{dt} = 0.2P - 0.002P^2$,

$\frac{dP}{dt} = 0.002P(100-P)$

here t is measured in weeks, $t \geq 0$.

a) If $P(0) = 5$, what is $\lim_{t\to\infty} P(t)$? 100
b) If $P(0) = 60$, what is $\lim_{t\to\infty} P(t)$? $= 100$
c) If $P(0) = 120$, what is $\lim_{t\to\infty} P(t)$? $= 100$
d) Sketch the solution curves for a), b), and c). Which one has an inflection point? (a) only

$P(t) = \frac{L}{1+Ce^{-kLt}}$

$$P(t) = \frac{3000}{1+Ce^{-3t}}$$

at $(0,50)$: $50 = \frac{3000}{1+C}$

$1+C = \frac{300\cancel{0}}{5\cancel{0}}$

$1+C = 60$

$\boxed{C = 59}$

$P(t) = \boxed{\frac{3000}{1+59e^{-3t}}}$

(b) $P(2) = \frac{3000}{1+59e^{-6}}$ ppl

(c) $1500 = \boxed{\frac{3000}{1+59e^{-3t}}}$

Example 8:

The rate at which the flu spread

$\frac{dP}{dt} = 0.001P(3000-P)$, where

a) If $P(0) = 50$, solve for
b) Use your solution to a)
c) Use your solution to a) fastest.

(a) $P(t) = \frac{L}{1+Ce^{-kLt}}$

$P(t) = \frac{3000}{1+Ce^{-3t}}$

$(0,50)$: $50 = \frac{3000}{1+C}$

$1+C = \frac{300\cancel{0}}{5\cancel{0}}$

$1+C = 60$

$\boxed{C=59}$

So, $\boxed{P(t) = \frac{3000}{1+59e^{-3t}}}$

(b) $P(2) = \frac{3000}{1+59e^{-6}}$ ppl