CHAPTER 9
SEQUENCES & SERIES

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + \cdots + u^n + \cdots \qquad -1 < u < 1$$

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + \cdots \qquad -\infty < u < +\infty$$

$$\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + (-1)^n \frac{u^{2n}}{(2n)!} + \cdots \qquad -\infty < u < +\infty$$

$$\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} + \cdots \qquad -\infty < u < +\infty$$

$$\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n} = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n+1} \frac{u^n}{n} + \cdots \qquad -1 < u \le 1$$

$$(1+u)^a = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} u^n \qquad -1 < u < 1$$

Convergence and Divergence

LAST UNIT OF CALCULUS!!!!!

A sequence is simply list of things generated by a rule

More formally, a **sequence** is a function whose domain is the set of positive integers, or **natural numbers**, n, such that $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The range of the function are called the terms in the sequence,

$$a_1, a_2, a_3, \ldots, a_{n,\ldots}$$

Where a_n is called the ***n*th term** (or rule of sequence), and we denote the sequence by $\{a_n\}$.

The sequence can be expressed by either

1) an ample number of terms in the sequence, separated by commas
2) an explicit function defined by the **rule of sequence**
3) the rule of sequence set off in braces.

The sequence $2, 4, 6, 8, \ldots$ is the sequence of even numbers. Express the same sequence as a rule of a non-negative integer n. The sequence $1, 3, 5, \ldots$ is the sequence of odd numbers. Express the same sequence as a rule of a non-negative integer n. How many in the list are needed to establish the "rule" in the absence of the explicitly-stated rule?

***NOTE: When given a sequence as a list, the first term is usually designated to be associated with $n = 1$. This is because we are using n as an ordinal (or counting) number, rather than a cardinal number.

We will be primarily interested in what happens to the sequence for increasingly large values of n.

Example 2:

If $a_n = \left\{ \frac{4n}{3+2n} \right\}$, list out the first five terms, then estimate $\lim_{n\to\infty} a_n$.

Let $\{a_n\}$ be a sequence of real numbers.

Possibilities:

1) If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity
2) If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
3) If $\lim_{n\to\infty} a_n = c$, an finite real number, then $\{a_n\}$ converges to c
4) If $\lim_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

Definition:

$n!$ is read as "n factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

Por ejemplo: $9! = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

Determine whether the following sequences converge or diverge.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

(c) $a_n = 3 + (-1)^n$

(d) $a_n = \frac{n}{1-2n}$

(e) $a_n = \frac{\ln n}{n}$

(f) $a_n = \frac{n!}{(n+2)!}$

(g) $a_n = \frac{2n!}{(n-1)!}$

(h) $a_n = \frac{n + (-1)^n}{n}$

(i) $a_n = \frac{(-1)^n (n-1)}{n}$

(j) $a_n = \dots (n+1)!$ (k) $a_n = \left(1 + \dots n\right)$ (l) $\left\{ \dots n^n \right\}$

Example 4:
Write an expression for the *n*th term.

(a) 3, 8, 13, 18, . . .

(b) 5, -15, 45, -135, . . .

(c) 1, 4, 9, 16, 25, . . .

(d) 4, 10, 28, 82, . . .

(e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \dots$

(f) $\ln 1,\ \ln 2,\ \ln 4,\ \ln 8, \ldots$

adding any number of terms from a sequence together: $a_1 + a_2 + a_3 + \cdots$. A series can be written more succinctly by using the summation symbol sigma, $\sum$., the Greek letter "S" for **Esum** (the "E" is both silent and not really there.)

For infinite series, we can look at the sequence **of partial sums**, that is, looking to see what the sums are doing as we add additional terms. In general, the *n*th partial sum of a series is denoted S_n. This can be explored on a calculator by adding sequential terms to the aggregate sum.

Example 5:

For both $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$, generate the sequence of partial sums S_1, S_2, S_3, …, S_n, for each, then determine if the sequences converges or diverges. Do the results surprise you? Where else have we seen something like this before?