Tests for Convergence and Divergence

Geometric Series, nth Term Test for Divergence, and Telescoping Series

Geometric Series Test (GST)

A geometric series is in the form $\sum_{n=0}^{\infty} a\cdot r^n$ or $\sum_{n=1}^{\infty} a\cdot r^{n-1}$, $a \neq 0$

The geometric series **diverges** if $|r| \geq 1$.

If $|r| < 1$, the series **converges** to the sum $S = \dfrac{a_1}{1-r}$.

Where a_1 is the first term, regardless of where n starts, and r is the common ration.

Example 8:
Using the GST, determine whether the following series converge or diverge. If the converge, find the sum.

(a) $\sum_{n=1}^{\infty} \dfrac{3}{2^n}$ (b) $\sum_{n=1}^{\infty} \left(\dfrac{3}{2}\right)^n$ (c) $\sum_{n=2}^{\infty} 3\left(-\dfrac{1}{2}\right)^n$

***n*th Term Test for Divergence (ONLY)**

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(think about it, it should make perfect sense!)

Note: This does **NOT** say that if $\lim_{n\to\infty} a_n = 0$, then the series **DOES** converge. This test can only be used to prove that a series diverges (hence the name.) If $\lim_{n\to\infty} a_n = 0$, then this test doesn't tell us anything, is inconclusive, doesn't work, fails, etc. . . . We MUST use another test. This test can be a GREAT time-saver. Always perform it **FIRST**, not second, but FIRST!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

Example 9:
Determine whether the following series converge or diverge. If they converge, find their sum.

(a) $\sum_{n=1}^{\infty} \dfrac{2n+3}{3n-5}$ (b) $\sum_{n=1}^{\infty} \dfrac{n!}{2n!+1}$ (c) $\sum_{n=1}^{\infty} \dfrac{3^n-2}{3^n}$ (d) $\sum_{n=2}^{\infty} \dfrac{1}{(1.1)^n}$

A series such as $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$ is called a **telescoping series** because it collapses to one term or just a few terms. If a series collapses to a finite sum, then it converges by the **Telescoping Series Test**.

Example 10:
Determine whether the following series converges or diverges. If they converge, find their sum.

(a) $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$ (b) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ (c) $\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3}$

Integral Test and *p*-Series

Integral Test
If *f* is **D**ecreasing, **C**ontinuous, and **P**ositive (**D**ogs **C**uss in **P**rison!) for $x \geq 1$ AND $a_n = f(x)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either BOTH converge or diverge.

Note 1: This does NOT mean that the series converges to the value of the definite integral!!!!!!
Note 2: The function need only be decreasing for all $x > k$ for some $k \geq 1$.

If the series converges to *S*, then the remainder, $R_n = |S - S_n|$ is bounded by $0 \leq R_n \leq \int_n^{\infty} f(x)dx$. (Not on AP exam, but on my exam.). This means $S \in [S_n, S_n + R_n]$.

Example 11:
Determine whether the following series converge or diverge. If they converge, find an interval in which the sum resides using S_4.

(a) $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ (b) $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

***p*-series**

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ is called a *p*-series, where *p* is a positive constant.

For $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ is called the **harmonic series**.

Based on your experience with *p*-series and their reliance on the number one, fill in chart below.

***p*-Series Test**

The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$, based on above

a) If $p = 1$, ____

b) If $p < 1$, ____

c) If $p > 1$, ____

Note: If the *p*-series converges, we cannot find its sum ☹ . This is more often the case than not.

Example 13:

Determine of the following converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$ (d) $\sum_{n=1}^{\infty} \frac{999999999}{n^{1.0000000001}}$

Based on your experience with improper integrals, again, fill in the chart below.

Comparison of Series

Direct Comparison Test (DCT)

If $a_n \ge 0$ and $b_n \ge 0$,

1) If $\sum_{n=1}^{\infty} b_n$ converges and $0 \le a_n \le b_n$, then $\sum_{n=1}^{\infty} a_n$ __________________________.

2) If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \le a_n \le b_n$, then $\sum_{n=1}^{\infty} b_n$ __________________________.

NOTE:You must state/show the inequality when stating the conclusion of the test!!

Example 14:

Determine whether the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

(c) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

(d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

(e) $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

(f) $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$