Taylor Polynomials and Approximations

Polynomial functions can be used to approximate other elementary functions such as $\sin x$, e^x, and $\ln x$.

Example 1:

Find the equation of the tangent line for $f(x) = \sin x$ at $x = 0$, then use it to approximate $\sin(0.2)$. Is this an over or an under approximation of $\sin(0.2)$?

The equation of the tangent line used in Ex. 1 is called a **first-degree Taylor polynomial**. Taylor polynomials of higher degree can be used to obtain increasingly better approximations of non-polynomial

On your calculator graph $y1 = \sin x$. Use the following window: $X[-9,9]$, $Y[-4,4]$. Now in $y2 =$, graph successively, adding an extra term each time, the following: $y2: x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

What do you notice? What is $y1(0)$? $y2(0)$? What is $y1(0.2)$? $y2(0.2)$?

If f has n derivatives at $x = c$, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the <u>nth-degree Taylor polynomial for f centered at c</u>, named after Brook Taylor, an English mathematician.

Note 1: A first-degree Taylor polynomial is a tangent line to f at c.

Note 2: $\frac{f^{(n)}(c)}{n!}$ **is the coefficient of the** $(x-c)^n$ **term**

If $c = 0$, then $P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$ is called the <u>nth-degree Maclaurin polynomial for f</u>, named after Scottish mathematician, Colin Maclaurin.

Note 3: Maclaurin not only got his name on a very specific case of Taylor's work, but he also had big hair.

Note 4: For Taylor & Mac Polynomials, you MUST use a "squiggle." For example, $f(x) \approx P_n(x) = \cdots$

Example 3:

Find the Maclaurin polynomial of degree $n = 5$ for $f(x) = \sin x$. Then use $P_5(x)$ to approximate the value of $\sin(0.1)$ using correct notation. Find the error for your approximation and determine an interval in which $\sin(0.1)$ could actually live. Finally, compare your approximation to the actual value of $\sin(0.1)$. Is it in your interval? Cool, huh?!

Example 4:

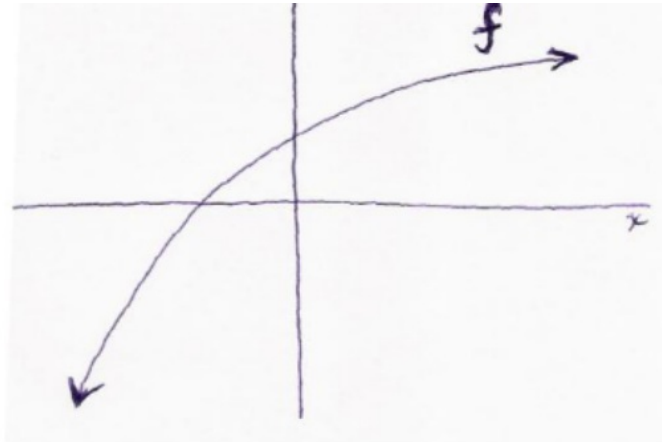
Find the Taylor polynomial of degree $n=6$ for $f(x)=\ln x$ at $c=1$. Then use $P_6(x)$ to approximate the value of $\ln(1.1)$

Example 5:

Suppose that g is a function which has continuous derivatives, and that $g(2) = 3$, $g'(2) = -4$, $g''(2) = 7$, $g'''(2) = -5$. Write the Taylor polynomial of degree 3 for g centered at 2.

Use a third-degree Taylor approximation of e^x for x near 0 to find $\lim_{x \to 0} \frac{e^x - 1}{2x}$, then compare it to the actual limit at zero.. Is this surprising? Why or why not.

Given that $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for f about $x = 0$, what can you say about the signs of a, b, and c if f has the graph pictured at the right? Justify your answer.



$f(x)=e^x$, the find the following Maclaurin Polynomials.

(a) $g(x)=\sin(2x)$

(b) $g(x)=x\cos(x)$

(c) $g(x)=4e^{x^2}$