9.3 Power Series: Taylor and MacLauren Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a **power series** (centered at $x = 0$).

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \cdots + a_n (x-c)^n + \cdots$$

is a power series **centered at** $x = c$, where c is a constant.

For a power series centered at $x = c$, exactly one of the following is true:

1) The series converges only at $x = c$. (ALL power series converge at their center!!)
2) The series converges for all x.
3) There exists an $R > 0$ such that the series converges for $|x - c| < R$ and diverges for $|x - c| > R$.

R is called the **radius of convergence** of the power series.

In part 1) the radius is 0.
In part 2), the radius is ∞
In part 3) The corresponding domain, $[(c - R, c + R)]$, is called the **interval of convergence** or the **<u>domain</u>** of the power series.

Note: to determine if the endpoints are included or not, we must test each endpoint independently.
Note2: We typically use the RATIO TEST to determine the radius of convergence.

Example 1:

Based on the fact that a 4th degree Maclaurin polynomial for $f(x)=e^x$ is $M_4(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$, find the *n*th term, then find the radius and interval of convergence for the representative power series.

Often, we will be dealing with power series representing unknown functions. While we may not recognize the function the series actually represents, we can still determine the values of x for which it ***does*** represent the unknown function.

Example 2:

Find the radius of convergence and the interval of convergence. Be sure to test the endpoints independently.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n2^n}$

(b) $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(d) $\sum_{n=0}^{\infty} n!(x-3)^n$

We will now look at a special family of power series for which you're practically already acquainted: **Taylor and Maclaurin Series**.

Taylor Series centered at $x = c$:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Once again, if $c = 0$, the series is called a **Maclaurin series**.

Example 3:

Find a Taylor series for $f(x) = e^{5x}$ centered at $c = 2$. Give the first four nonzero terms and the general term.

There are four special Maclaurin series you must know. These are the series for e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$. These series, under the operations of calculus, behave like the functions they represent on their interval of convergence. For series with a finite interval of convergence, such as the series for $\frac{1}{1-x}$, taking the derivative or integral **will not change the radius** of convergence **but may change the endpoints of the interval** of convergence.

Once we have these series memorized these series, we can conveniently manipulate them to suit other similar non-polynomial functions as we did in the previous section with Taylor polynomials.

You can manipulate these three special series (or any series we are given) to find other series by using the following techniques. Note: the radius of convergence may change, though)

1) Substitute into a series for x
2) Multiply or divide the series by a constant and/or a variable
3) Add or subtract two series
4) Differentiate or integrate a series (may change the interval, but not the radius of convergence)
5) **Recognize the series as the sum of a geometric power series (next section)**

Example 4:

If $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$, Find $f'(x)$ and $f'(0)$. Do you recognize this familiar function?